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On the existence of components of the Hilbert scheme with the expected number of moduli

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1 Introduction

The purpose of this paper is to investigate, for given r, n, g, the existence of a smooth irreducible open subset of the Hilbert scheme of $\mathbb{P}^r$ parametrizing nonsingular irreducible curves of degree n and genus g having the "expected number of moduli." Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g, and, for $r \geqq 3$, let $H_{n,g,r}$ be the closure in $\mathrm{Hilb}\,\mathbb{P}^r$ of the open set parametrizing nonsingular irreducible curves of degree n and genus g and let $\varrho = \varrho(g, n, r) = g - (r+1)(g-n+r)$ be the Brill-Noether number. Following Sernesi [S], we give the ensuing

(1.1) **Definitions.** An irreducible component W of $H_{n,g,r}$ is said to have the *expected number of moduli* if the image of the rational functorial map

$$\pi : W \to \mathcal{M}_g$$

has dimension $\min\{3g-3, 3g-3+\varrho\}$. W is said to be *regular* if $H^1(N_C) = 0$ for C a general curve in W.

If $\varrho \geqq 0$ it is a consequence of Brill-Noether theory that there is a unique component of $H_{n,g,r}$ dominating $\mathcal{M}_g$, i.e., having the expected number of moduli. On the other hand, for $\varrho \leqq 0$, very little about existence or uniqueness of such components is known. Sernesi in [S] proved that there exists a regular component of $H_{n,g,r}$ having the expected number of moduli for any r, n, g such that $r \geqq 3$, $n \geqq r+1$ and $n - r \leqq g \leqq \dfrac{r(n-r)-1}{r-1}$. Recent improvements of this result have been obtained for the existence by Eisenbud and Harris (announced in [EH]), Ballico and Ellia [BE] and, in the case $r = 3$ for the existence (and nonuniqueness, in some cases) by Pareschi [P].

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In this paper we want to generalize (for $r \geqq 4$) the mentioned results by extending the range of possible r, n, g such that there exists a regular component of $H_{n,g,r}$ with the expected number of moduli to $-\frac{5}{4}(g-r^2) \leqq \varrho \leqq 0$ (asymptotically in r). More precisely, for any $r \geqq 3$, define ε by $\frac{r-\varepsilon}{3} = \left\lceil \frac{r-1}{3} \right\rceil$ and $f(r) = (15r^3 + 5r^2 + 3\varepsilon r^2 - 31r + 11\varepsilon r + 2\varepsilon^2 - 8\varepsilon + 9)/(12r + 27 - 3\varepsilon)$ (observe that $f(r)$ is asymptotically like $\frac{5}{4}r^2$). Then we will show:

(1.2) **Theorem.** *There exists a regular component of $H_{n,g,r}$ with the expected number of moduli for any r, n, g satisfying:*

either

$$r \leqq 9 \text{ or } r = 11 \text{ and } -\max\left\{\frac{1}{r}g - \frac{r+1}{r}, g - r^2 - 1\right\} \leqq \varrho = \varrho(g, n, r) \leqq 0$$

or

$$r \geqq 12 \text{ or } r = 10 \text{ and } -\max\left\{\frac{1}{r}g - \frac{r+1}{r}, g - r^2 - 1, \frac{5r+\varepsilon}{4r+9-\varepsilon}g - f(r)\right\} \leqq \varrho \leqq 0.$$

(1.3) *Remark.* The problem of existence of components of $H_{n,g,r}$ with the expected number of moduli makes sense for $\varrho \geqq -3g+3$ (for $g \geqq 2$). Let us define $\varrho_{\min}(g, r) = \inf\{k \geqq -3g+3$: there exists a component of $H_{n,g,r}$ with the expected number of moduli for some n such that $\varrho(g, n, r) = k\}$. It has been proved so far (for simplicity we will state this only for $g \gg r$) that

$$\varrho_{\min}(g, r) \leqq \begin{cases} -3g + 8\sqrt[3]{9}g^{2/3} - 12 & \text{for } r = 3 \text{ [P]} \\ -\frac{7}{3}g & \text{for } r = 4 \text{ [BE]} \\ -\left(1 + \frac{2}{r}\right)g & \text{for } 5 \leqq r \leqq 17 \text{ [BE]} \\ -\frac{5r+\varepsilon}{4r+9-\varepsilon}g + f(r) & \text{for } r \geqq 18 \text{ (by Theorem 1.2).} \end{cases}$$

It seems reasonable to expect the optimal result to be $\varrho_{\min}(g, r) = -3g + o(g)$. For $r = 3$ this is true (by [P]) and is easily seen to be the best possible: In fact the Castelnuovo bound already implies $\varrho(g, n, 3) \geqq -3g + 8\sqrt{g} - 4$. Moreover, as we will see in Example 4.3, it is actually $\varrho_{\min}(g, 3) \geqq -3g + 2\sqrt{24g-15} - 6$. For $r \geqq 4$ instead the inequality $\varrho \geqq -3g+3$ only means that g is at most linear in n and therefore does not seem to put strong restrictions to smooth curves of degree n and genus g. On the other hand it is worth to observe that we do not expect (for $g \geqq 2$) the equality $\varrho_{\min}(g, r) = -3g+3$ to hold, because this would mean the existence of a component of $H_{n,g,r}$ whose image in $\mathcal{M}_g$ is a point.

We will work in the category of schemes over an algebraically closed field of characteristic 0. By divisor we will mean Cartier divisor; if D is a divisor on a scheme X and $\mathcal{F}$ a sheaf on X, we will write

$$H^i(\mathcal{F}), h^i(\mathcal{F}), \mathcal{F}(D)$$

instead of $H^i(X, \mathcal{F})$, $\dim H^i(X, \mathcal{F})$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ respectively.

2 A property of general nonspecial curves

In this section we will prove some vanishing of cohomology groups of the normal and restricted tangent bundle of nonspecial curves twisted by a suitable divisor. These vanishings will provide the key tool for the proof of Theorem 1.2.

For $r \geqq 3$ let $\Gamma \subset \mathbb{P}^r$ be a smooth irreducible nondegenerate (i.e., not contained in a hyperplane) linearly normal nonspecial [i.e., $H^1(\mathcal{O}_\Gamma(1))=0$] curve. By Proposition 3.3 of [S] and Lemma 4.1 below it is clear that any component W of Hilb$\mathbb{P}^r$ containing Γ must dominate $\mathcal{M}_g$ [since $T_{\mathbb{P}^r|\Gamma}$ and N_Γ are both quotients of $\mathcal{O}_\Gamma(1)^{r+1}$]. But $\varrho(g(\Gamma), \deg(\Gamma), r) \geqq 0$ hence W is unique (see Sect. 1). We will use this fact in the proof of the proposition below. Let $P_1, \ldots, P_{r+4}$ be $r+4$ general points in Γ and denote by $D=P_1+\ldots+P_{r+4}$ their divisor. Then:

(2.1) **Proposition.** *Let N_Γ be the normal bundle of Γ in $\mathbb{P}^r$ and g the genus of Γ. If Γ as above is general in its component of* Hilb$\mathbb{P}^r$ *and $g \geqq \frac{r-1}{3}$ we have*

(i) $H^1(N_\Gamma(-D))=0$;

(ii) $H^0(T_{\mathbb{P}^r|\Gamma}(-D))=0$ *for* $g \leqq r$.

Proof. To prove (ii) we degenerate Γ into a union of two rational normal curves. Let $X \subset \mathbb{P}^g$ and $Y \subset \mathbb{P}^r$ be two rational normal curves meeting transversally in $g+1$ points. Let Δ be the divisor of Y given by these points, D_X and D_Y be two divisors on X and Y of degrees $g+3$ and $r+1-g$ respectively. Let $\Gamma' = X \cup Y$ and $D' = D_X + D_Y$ the divisor on Γ'. Then we have an exact sequence

$$0 \to T_{\mathbb{P}^r|Y}(-\Delta - D_Y) \to T_{\mathbb{P}^r|\Gamma'}(-D') \to T_{\mathbb{P}^r|X}(-D_X) \to 0$$

so it will be enough to show

$$H^0(T_{\mathbb{P}^r|Y}(-\Delta - D_Y)) = H^0(T_{\mathbb{P}^r|X}(-D_X)) = 0. \tag{2.2}$$

From (2.2) it will follow $H^0(T_{\mathbb{P}^r|\Gamma'}(-D'))=0$, hence (ii) by semicontinuity since $\deg D' = g+3+r+1-g = r+4$. But $T_{\mathbb{P}^r|Y} \cong \mathcal{O}_{\mathbb{P}^1}(r+1)^r$ and $T_{\mathbb{P}^r|X} \cong \mathcal{O}_{\mathbb{P}^1}(g+1)^g \oplus \mathcal{O}_{\mathbb{P}^1}(g)^{r-g}$ so

$$T_{\mathbb{P}^r|Y}(-\Delta - D_Y) \cong \mathcal{O}_{\mathbb{P}^1}(r+1-g-1-r-1+g)^r = \mathcal{O}_{\mathbb{P}^1}(-1)^r$$

and

$$\begin{aligned} T_{\mathbb{P}^r|X}(-D_X) &\cong \mathcal{O}_{\mathbb{P}^1}(g+1-g-3)^g \oplus \mathcal{O}_{\mathbb{P}^1}(g-g-3)^{r-g} \\ &= \mathcal{O}_{\mathbb{P}^1}(-2)^g \oplus \mathcal{O}_{\mathbb{P}^1}(-3)^{r-g}. \end{aligned}$$

Therefore (2.2) is true and (ii) is proved. Now (i) follows by semicontinuity from

(2.3) **Lemma.** *For every $g \geqq \frac{r-1}{3}$ there is a linearly normal nonspecial curve Γ lying on a rational normal surface scroll $S_{r-1} \subset \mathbb{P}^r$ such that*

$$H^1(N_{\Gamma/S}(-D)) = H^1(N_{S|\Gamma}(-D)) = 0,$$

where D is a divisor on Γ sum of $r+4$ general points.

Proof. Let $e \geqq 0$ and $n > e$ such that $r-1 = 2n-e$ and let S be the embedding of the ruled surface $X_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ via the linear system $|C_0 + nf|$ where f is a fiber and C_0 is a section of $X_e \to \mathbb{P}^1$ with $\mathcal{O}_{X_e}(C_0) = \mathcal{O}(1)$. Let C_1 and Γ be general curves in the linear systems $|C_0+(g+1+e)f|$ and $|C_0+C_1|=|2C_0+(g+1+e)f|$ respectively. Assume now n and e are chosen so that

$$\begin{cases} e\leqq g+1 \\ g+1+n\geqq r. \end{cases} \tag{2.4}$$

Then C_0 is a rational normal curve of degree $n-e$ in a $\mathbb{P}^{n-e}$ and C_1 is a smooth nondegenerate rational curve of degree $g+1+n$ in $\mathbb{P}^r$ (by (2.4)). Moreover, $C_0\cdot C_1 = -e+g+1+e=g+1$ so $p_a(\Gamma)=g+1-2+1=g$ and $\deg\Gamma=g+1+n+n-e=g+r$.

Finally, if $\Delta=C_0\cap C_1$ and $\Gamma'=C_0\cup C_1$ we have an exact sequence

$$0\to\mathcal{O}_{C_1}(1)(-\Delta)\to\mathcal{O}_{\Gamma'}(1)\to\mathcal{O}_{C_0}(1)\to 0,$$

where

$$\mathcal{O}_{C_1}(1)(-\Delta)\cong\mathcal{O}_{\mathbb{P}^1}(g+1+n-g-1)=\mathcal{O}_{\mathbb{P}^1}(n)$$
$$\mathcal{O}_{C_0}(1)\cong\mathcal{O}_{\mathbb{P}^1}(n-e),$$

hence $H^1(\mathcal{O}_{C_1}(1)(-\Delta))=H^1(\mathcal{O}_{C_0}(1))=0$ and therefore Γ' is linearly normal nonspecial and so is Γ. Note also that by (2.4), Γ is smooth irreducible [H, V, Corollary 2.18]. First we show that

$$H^1(N_{\Gamma/S}(-D))=0. \tag{2.5}$$

To this end it would be enough to find two effective divisors D_0 on C_0, D_1 on C_1 of degrees δ_0 and δ_1 respectively, such that

$$\delta_0+\delta_1=r+4$$

and

$$H^1(\mathcal{O}_{C_0}(-D_0)\otimes\mathcal{O}_S(\Gamma'))=H^1(\mathcal{O}_{C_1}(-D_1-\Delta)\otimes\mathcal{O}_S(\Gamma'))=0. \tag{2.6}$$

In fact, if we let $D'=D_0+D_1\subset\Gamma'$, we have an exact sequence

$$0\to\mathcal{O}_{C_1}(-D_1-\Delta)\otimes\mathcal{O}_S(\Gamma')\to N_{\Gamma'/S}(-D')\to\mathcal{O}_{C_0}(-D_0)\otimes\mathcal{O}_S(\Gamma')\to 0$$

so $H^1(N_{\Gamma'/S}(-D'))=0$ and therefore also $H^1(N_{\Gamma/S}(-D))=0$ by semicontinuity. Now

$$\mathcal{O}_{C_0}(-D_0)\otimes\mathcal{O}_S(\Gamma')\cong\mathcal{O}_{\mathbb{P}^1}(-\delta_0-e+g+1)$$

and

$$\mathcal{O}_{C_1}(-D_1-\Delta)\otimes\mathcal{O}_S(\Gamma')\cong\mathcal{O}_{\mathbb{P}^1}(-\delta_1-g-1+g+1+2g+2+e)$$
$$=\mathcal{O}_{\mathbb{P}^1}(2g+2+e-\delta_1).$$

So we just need δ_0 and δ_1 such that: $\delta_0\geqq 0$, $\delta_1\geqq 0$; $\delta_0+\delta_1=r+4$, $-\delta_0-e+g+1\geqq -1$, $2g+2+e-\delta_1\geqq -1$. For example we can choose δ_0 and δ_1 as follows:

$$\text{if } e\leqq g-r-2 \text{ let } \delta_0=r+4, \quad \delta_1=0;$$
$$\text{if } e>g-r-2 \text{ let } \delta_0=g+2-e, \quad \delta_1=r+4-\delta_0$$

$\left(\text{note that } 2g+2+e-\delta_1\geqq -1 \text{ since } g\geqq \frac{r-1}{3}\right)$. This proves (2.6) and hence (2.5).

We turn now to the proof of

$$H^1(N_{S|\Gamma}(-D))=0. \tag{2.7}$$

With notation as above, (2.7) follows from

$$H^1(N_{S|C_1}(-\Delta-D_1))=H^1(N_{S|C_0}(-D_0))=0. \tag{2.8}$$

In fact, the exact sequence

$$0\to N_{S|C_1}(-\Delta-D_1)\to N_{S|\Gamma'}(-D')\to N_{S|C_0}(-D_0)\to 0$$

shows that $H^1(N_{S|\Gamma'}(-D'))=0$ and hence (2.7) by semicontinuity. To show (2.8), let $\delta_0=n-e+2$ and $\delta_1=n+3$. From the exact sequence

$$0\to N_{C_1/S}(-\Delta-D_1)\to N_{C_1/\mathbb{P}^r}(-\Delta-D_1)\to N_{S|C_1}(-\Delta-D_1)\to 0$$

we see that the first vanishing in (2.8) follows from

$$H^1(N_{C_1/\mathbb{P}^r}(-\Delta-D_1))=0. \tag{2.9}$$

Since $C_1\cong\mathbb{P}^1$ we know [Sa, Proposition 1] that $N_{C_1/\mathbb{P}^r}\cong\bigoplus_{i=1}^{r-1}\mathcal{O}_{\mathbb{P}^1}(b_i)$ with $b_i\geqq\deg C_1+2$. Hence

$$N_{C_1/\mathbb{P}^r}(-\Delta-D_1)\cong\bigoplus_{i=1}^{r-1}\mathcal{O}_{\mathbb{P}^1}(b_i-g-1-\delta_1),$$

so (2.9) is true since $b_i-g-1-\delta_1\geqq\deg C_1+2-g-1-n-3=g+1+n+2-g-1-n-3=-1$, $\forall i=1,\dots,r-1$. Let $N_{S|C_0}\cong\bigoplus_{i=1}^{r-2}\mathcal{O}_{\mathbb{P}^1}(a_i)$; then to finish the proof of (2.8) we need $a_i-\delta_0\geqq-1$, that is

$$a_i\geqq n-e+1, \quad \forall i=1,\dots,r-2. \tag{2.10}$$

Observe that $N_S^\vee(1)_{|C_0}\cong\bigoplus_{i=1}^{r-2}\mathcal{O}_{\mathbb{P}^1}(-a_i+n-e)$, hence (2.10) is equivalent to

$$H^0(N_S^\vee(1)_{|C_0})=0. \tag{2.10$'$}$$

From the exact sequence

$$0\to N_S^\vee(1)_{|C_0}\to N^\vee_{C_0/\mathbb{P}^r}(1)\to N^\vee_{C_0/S}(1)\to 0$$

we get $H^0(N_S^\vee(1)_{|C_0})=\ker\{\varphi: H^0(N^\vee_{C_0/\mathbb{P}^r}(1))\to H^0(N^\vee_{C_0/S}(1))\}$, so we have to show that φ is injective. Now

$$H^0(N^\vee_{C_0/\mathbb{P}^r}(1))=H^0((\mathcal{I}_{C_0}/\mathcal{I}^2_{C_0})(1))\cong H^0(\mathcal{I}_{C_0}(1)) \text{ and}$$
$$H^0(N^\vee_{C_0/S}(1))=H^0((\mathcal{I}_{C_0/S}/\mathcal{I}^2_{C_0/S})(1))\cong H^0(\mathcal{I}_{C_0/S}(1))$$
$$=H^0(\mathcal{O}_S(nf))\cong H^0(\mathcal{O}_{\mathbb{P}^1}(n))$$

so if H is a hyperplane containing C_0, then $H\cap S=C_0\cup C$, $C\sim nf$ and

$$\varphi(H)=\text{divisor on } \mathbb{P}^1\cong C_0 \text{ given by } C \text{ on } C_0.$$

By the projective characterization of S, we can find [ACGH, p. 96] coordinates on $\mathbb{P}^r$ so that the ideal of S is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1\dots x_{n-e-1} & \vdots & x_{n-e+1}\dots x_{r-1}\\ x_1 & x_2\dots x_{n-e} & \vdots & x_{n-e+2}\dots x_r\end{pmatrix}=(A,B),$$

where A is the matrix whose minors generate the ideal of $C_0\subset\mathbb{P}^{n-e}$ and B is the matrix whose minors generate the ideal of a rational normal curve $Y\subset\mathbb{P}^n$, and S is obtained by choosing an isomorphism $C_0\cong Y\cong\mathbb{P}^1$ and joining with lines the points of C_0 and Y corresponding in the isomorphism. Therefore, if we parametrize $Y \subset \mathbb{P}^r$ with $x_0 = \ldots = x_{n-e} = 0$, $x_{n-e+1+i} = s^{n-i}t^i$, $i = 0, \ldots, n$, $[s,t] \in \mathbb{P}^1$ and we observe that $H^0(\mathscr{I}_{C_0}(1))$ is generated by $x_{n-e+1}, \ldots, x_r$, then φ is given by

$$\varphi(x_{n-e+1+i}) = s^{n-i}t^i$$

so it is injective (in fact, φ is an isomorphism).

This proves (2.10)' and hence to conclude the proof of Lemma 2.3 we need to find for every $r \geqq 3$ and $g \geqq \frac{r-1}{3}$, integers n and e so that (2.4) is satisfied.

(2.11) *Claim.* To satisfy (2.4) we can choose $e = 2n - r + 1$ and

$$n = \begin{cases} r-2 & \text{if } \; g \geqq r-4 \\ \dfrac{g+r}{2} & \text{if } \; g+r \equiv 0 \,(\mathrm{mod}\, 2) \text{ and } g < r-4 \\ \dfrac{g+r-1}{2} & \text{if } \; g+r \equiv 1 \,(\mathrm{mod}\, 2) \text{ and } g < r-4 . \end{cases}$$

Proof. The inequalities of (2.4) are equivalent to $n \geqq r-1-g$ and $n \leqq \frac{g+r}{2}$, respectively. Moreover, $e \geqq 0$ gives $n \geqq \frac{r-1}{2}$ and $n > e$ is $n \leqq r-2$. So any n satisfying

$$\max\left\{\frac{r-1}{2}, r-1-g\right\} \leqq n \leqq \min\left\{r-2, \frac{g+r}{2}\right\}$$

will do. Since $r \geqq 3$ and $g \geqq \frac{r-1}{3}$, the choice in Claim 2.11 satisfies this. Therefore Claim 2.11 is proved and so is Lemma 2.3.

3 A family of curves $C \subset \mathbb{P}^r$ with $H^1(N_C) = 0$, $h^0(T_{\mathbb{P}^r|C}) = (r+1)^2 - 1$

We will construct here, using Proposition 2.1 and some smoothing techniques, families of smooth curves in $\mathbb{P}^r$ with the properties indicated above. Let $C \subset \mathbb{P}^r$ be a nondegenerate curve and define the following properties:

Property (α): $H^1(N_C) = 0$.

Property (β): $h^0(T_{\mathbb{P}^r|C}) = (r+1)^2 - 1$.

(3.1) *Property* (γ): $\forall g: \frac{r-1}{3} \leqq g \leqq r$, there exist $r+4$ general points $P_1, \ldots, P_{r+4} \in \mathbb{P}^r$ and a nonspecial linearly normal smooth irreducible nondegenerate curve $\Gamma \subset \mathbb{P}^r$ of genus g with

$$H^1(N_\Gamma(-P_1 - \ldots - P_{r+4})) = H^0(T_{\mathbb{P}^r|\Gamma}(-P_1 - \ldots - P_{r+4})) = 0$$

such that there is a deformation of C meeting Γ quasitransversally in $P_1, \ldots, P_{r+4}$.

(3.2) *Remark.* Curves $\Gamma \subset \mathbb{P}^r$ as in Property (γ) exist by Proposition 2.1.

The construction of curves satisfying (α) and (β) will be inductive, starting with some families of curves implicitly contained in [S].

(3.3) **Lemma.** *For every $r \geqq 3$ and n, g such that*

$$-\max\left\{\frac{1}{r}g - \frac{r+1}{r},\, g - r^2 - 1\right\} \leqq \varrho = \varrho(g,n,r) \leqq 0,$$

there exists a smooth irreducible nondegenerate curve $C \subset \mathbb{P}^r$ of degree n, genus g satisfying properties (α), (β), and (γ) (the latter for $r \geqq 4$).

Proof. By [S, Theorem 6.1], we see that curves C as in the Lemma, satisfying (α) and (β), exist in the range

$$-\frac{1}{r}g + \frac{r+1}{r} \leqq \varrho \leqq 0, \quad n \geqq 2r \tag{3.4}$$

[(β) follows from Lemma 4.1 below].

To extend this range, for $n \geqq r^2 + r$, we use

(3.5) **Sublemma.** *Let $C \subset \mathbb{P}^r$ be a smooth irreducible nondegenerate curve of degree $n \geqq r+2$ (respectively $n \geqq r+1$) and genus g satisfying (α), (β), and (γ). Let $H \subset \mathbb{P}^r$ be a general hyperplane and $X \subset H$ a rational normal curve meeting C in $r+2$ points (respectively $r+1$ points). Then*

$$C' = C \cup X$$

is flatly smoothable and its general smoothings satisfy (α), (β), and (γ).

Proof. Let C'' be a general deformation of C'. The facts that C'' is smooth and satisfies (α) are already in [S, Theorem 5.2]; C'' also satisfies (γ) because C' does (since C does). To see (β), let $\Delta = C \cap X \subset X$ and consider the exact sequence

$$0 \to T_{\mathbb{P}^r|X}(-\Delta) \to T_{\mathbb{P}^r|C'} \to T_{\mathbb{P}^r|C} \to 0.$$

It will suffice to show that

$$H^0(T_{\mathbb{P}^r|X}(-\Delta)) = 0. \tag{3.6}$$

In fact this implies that

$$h^0(T_{\mathbb{P}^r|C'}) \leqq h^0(T_{\mathbb{P}^r|C}) = (r+1)^2 - 1 \quad \text{(by (β) for } C),$$

so, since $h^0(T_{\mathbb{P}^r|C''}) \leqq h^0(T_{\mathbb{P}^r|C'})$ by semicontinuity, we get (β) for C'' by Lemma 4.1. For (3.6) we observe that

$$T_{\mathbb{P}^r|X} \cong T_{\mathbb{P}^{r-1}|X} \oplus \mathcal{O}_X(1)$$

hence

$$T_{\mathbb{P}^r|X}(-\Delta) \cong \mathcal{O}_{\mathbb{P}^1}(r-r-2)^{r-1} \oplus \mathcal{O}_{\mathbb{P}^1}(r-1-r-2) = \mathcal{O}_{\mathbb{P}^1}(-2)^{r-1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$$

[respectively, $T_{\mathbb{P}^r|X}(-\Delta) \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{r-1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$] has no global sections. This proves Sublemma 3.5.

To get the range of Lemma 3.3, we apply repeatedly Sublemma 3.5 in the case of $r+2$ points to curves in range (3.4). This gives a transformation $(n,g) \mapsto (n+r-1, g+r+1)$, hence it is enough to produce, for every couple of integers (n,g) such that

$$n \geqq r^2 + r \quad \text{and} \quad -g + r^2 + 1 \leqq \varrho < -\frac{1}{r}g + \frac{r+1}{r},$$

an integer $i \geqq 0$ such that

$$n - (r-1)i \geqq 2r$$

and

$$-\frac{1}{r}[g-(r+1)i]+\frac{r+1}{r}\leqq\varrho(g-(r+1)i,n-(r-1)i,r)\leqq 0.$$

For example $i=\left[\frac{rg-n(r+1)+r(r+1)}{r+1}\right]$ (where [ ] denotes integer part) satisfies the above inequalities.

It remains to see that curves in range (3.4) satisfy (γ). To this end we recall Sernesi's construction [S, Theorem 6.1]: Curves C with (n,g) in range (3.4) are obtained by applying $g-n+r$ times Sublemma 3.5, in the case of $r+1$ points, to linearly normal nonspecial curves. That is, for every (n,g) in range (3.4), let $i=g-n+r$, $n'=n-(r-1)i$, $g'=g-ri$ and let $C_0\subset\mathbb{P}^r$ be a smooth irreducible nondegenerate linearly normal nonspecial curve of degree n' and genus g' (note that $n'=g'+r$, $g'\geqq 1$).

For every $j\geqq 1$ let C_j be a general smoothing of $C_{j-1}\cup X$ (where X is chosen as in Sublemma 3.5, in the case of $r+1$ points). Then $C=C_i$.

Therefore it is enough to see that C_1 satisfies (γ)

$$\left(\text{because } i=g-n+r\geqq\frac{r+1}{r}(n-r)-n+r=\frac{n-r}{r}\geqq\frac{2r-r}{r}=1\right).$$

(3.7) *Claim.* Let $r\geqq 4$ and $P_1,\ldots,P_{r+3}$ be $r+3$ general points of $Y\subset\mathbb{P}^r$ a smooth irreducible nondegenerate curve passing through them. Then there exists a smooth irreducible nondegenerate linearly normal nonspecial curve $C_0\subset\mathbb{P}^r$ of genus $g_0\geqq 1$ meeting Y quasi-transversally in $P_1,\ldots,P_{r+3}$.

Let us assume Claim 3.7 and prove that C_1 satisfies (γ). Choose $r+4$ general points $P_1,\ldots,P_{r+4}\in\mathbb{P}^r$; choose any Γ through them as in Property (γ). Claim 3.7 (with $Y=\Gamma$) gives a curve C_0 meeting Γ quasi-transversally in $P_1,\ldots,P_{r+3}$. Let H be a general plane through P_{r+4} and $X\subset H$ a rational normal curve meeting C_0 in $r+1$ points and passing through P_{r+4}. Then $C_1=$ a general smoothing of $C_0\cup X$ satisfies (γ).

Finally it is clear that Claim 3.7 will follow as soon as we prove it for $g_0=1$ (for $g_0\geqq 2$ one can attach g_0-1 general 2-secants to the curve of genus 1 and take a smoothing).

To this end, let $M=\langle P_1,\ldots,P_{r-1}\rangle\cong\mathbb{P}^{r-2}$ and $N=\langle P_r,P_{r+1}\rangle\cong\mathbb{P}^1$. The trisecant lemma [ACGH, p. 110] implies that $M\cap N=\emptyset$. Also $\dim\langle M,P_{r+2}\rangle=\dim\langle M,P_{r+3}\rangle=r-1$, hence there are two lines L_1 and L_2 through P_{r+2}, P_{r+3}, respectively, meeting M and N. Let Z be a rational normal curve in M passing through $P_1,\ldots,P_{r-1}$, $Q_1=L_1\cap M$, $Q_2=L_2\cap M$ and let $E=Z\cup L_1\cup L_2\cup N$. Now E is elliptic of degree $r+1$ and meets Y quasi-transversally in $P_1,\ldots,P_{r+3}$. In fact Z and N are quasi-transversal to Y and meet Y exactly in $P_1,\ldots,P_{r-1}$ and P_{r+1}, P_r, respectively (because a consequence of the trisecant lemma is that $k+1$ general points of Y span a $\mathbb{P}^k$ meeting Y exactly at those points, for $k\leqq r-2$). Moreover, L_1 (or L_2) is not tangent to Y, otherwise the tangent line in the generic point of Y would meet the generic 2-secant N (which would imply that Y is strange, and therefore a line [H, IV, Theorem 3.9]). Also L_1 (or L_2) does not meet Y in any other point, otherwise the $\mathbb{P}^2$ spanned by P_r, P_{r+1}, and P_{r+2} would meet Y in some other point. But this is not possible because P_r, P_{r+1}, P_{r+2} are general points of Y and $r\geqq 4$. Now take C_0 a general smoothing of E. This shows Claim 3.7 and therefore concludes the proof of Lemma 3.3.

Define the following functions:

$$(3.8) \qquad \varphi(r)=\frac{27r^3+44r^2-4r+8\varepsilon r+2\varepsilon^2-8\varepsilon+9}{3(r+1)(4r-\varepsilon)},$$

$$N(r)=\frac{3r^4-19r^3+6\varepsilon r^3-72r^2+17\varepsilon r^2+r+2\varepsilon^2 r-10\varepsilon r-2\varepsilon^2+11\varepsilon-9}{3(r+1)(r-9+2\varepsilon)}.$$

Then we have

(3.9) **Proposition.** *There exists a smooth irreducible curve $C\subset\mathbb{P}^r$ with*

(α) $H^1(N_C)=0$

(β) $h^0(T_{\mathbb{P}^r|C})=(r+1)^2-1$

and degree n, genus g, for every r, n, g such that $r\geqq 12$ *or* $r=10$ *and*

$$(3.10)\ \frac{r+1}{r}(n-r)\leqq g\leqq \max\left\{\frac{r(n-r)-1}{r-1},\frac{r+1}{r-1}n-\frac{2r^2+r+1}{r-1},\frac{4r+9-\varepsilon}{4r-\varepsilon}n-\varphi(r)\right\};$$

$r\leqq 9$ *or* $r=11$ *and*

$$(3.11) \qquad \frac{r+1}{r}(n-r)\leqq g\leqq \max\left\{\frac{r(n-r)-1}{r-1},\frac{r+1}{r-1}n-\frac{2r^2+r+1}{r-1}\right\}.$$

Proof. By Lemma 3.3, we need to show the existence only in the case $r\geqq 12$ or $r=10$, $n\geqq N(r)$ and

$$(3.12) \qquad \frac{r+1}{r-1}n-\frac{2r^2+r+1}{r-1}\leqq g\leqq \frac{4r+9-\varepsilon}{4r-\varepsilon}n-\varphi(r).$$

[Note that $N(r)$ is defined so that range (3.12) is not empty.] First we prove

(3.13) **Lemma.** *Let $C\subset\mathbb{P}^r$ be a curve satisfying (α), (β), and (γ) and $\Gamma\subset\mathbb{P}^r$ be a general nonspecial linearly normal curve of genus $\frac{r-\varepsilon}{3}$ meeting a general deformation $\bar{C}$ of C quasi-transversally at $r+4$ general points. Then $C'=\bar{C}\cup\Gamma$ is flatly smoothable and its general smoothings satisfy (α), (β), and (γ).*

Proof. Note first that the construction is possible since C satisfies (γ). Moreover, to show that $C'=\bar{C}\cup\Gamma$ is flatly smoothable, it is enough to prove that [S, Proposition 1.6]

$$(3.14) \qquad H^1(N'_{C'})=0 \quad \text{where} \quad N'_{C'}=\ker(N_{C'}\to T^1_{C'})$$

($T^1_{C'}$ being the first cotangent sheaf of C'). Let $\{P_1,\dots,P_{r+4}\}=\bar{C}\cap\Gamma$ and $D=P_1+\dots+P_{r+4}$ on Γ. From the exact sequences [S, Lemma 5.1]

$$0\to N_{C'}\otimes\mathcal{O}_\Gamma(-D)\to N'_{C'}\to N_{\bar{C}}\to 0$$
$$0\to N_\Gamma(-D)\to N_{C'}\otimes\mathcal{O}_\Gamma(-D)\to T^1_{C'}\to 0$$

we get (3.14) since

$H^1(N_\Gamma(-D))=0$ by Proposition 2.1 (i),

$H^1(T^1_{C'})=0$ because $T^1_{C'}$ is supported on $\bar{C}\cap\Gamma$ and

$H^1(N_{\bar{C}})=0$ since C satisfies (α).

The definition of $N'_{C'}$ and (3.14) also imply $H^1(N_{C'})=0$, hence (α) holds for a general smoothing of C' by semicontinuity.

To see (β), consider the exact sequence

$$0\to T_{\mathbb{P}^r|\Gamma}(-D)\to T_{\mathbb{P}^r|C'}\to T_{\mathbb{P}^r|\bar{C}}\to 0.$$

By Proposition 2.1(ii), we get

$$h^0(T_{\mathbb{P}^r|C'})\leqq h^0(T_{\mathbb{P}^r|\bar{C}})=(r+1)^2-1 \qquad (\text{by } (\beta) \text{ for } C),$$

hence (β) holds for any general smoothing of C' (again by using semicontinuity and Lemma 4.1 below). Moreover, C' satisfies (γ) (just take another Γ through $r+4$ general points of $\bar{C}$), hence so does a general smoothing. This proves Lemma 3.13.

To finish the proof of Proposition 3.9, we will see that the construction of Lemma 3.13 gives curves with r, n, g in range (3.12).

From Lemma 3.13 we get $\deg C'=\deg C+\frac{4r-\varepsilon}{3}$, $p_a(C')=g(C)+\frac{4r+9-\varepsilon}{3}$, hence, given (n,g) in range (3.12) a curve of degree n, genus g, satisfying (α) and (β) will exist as soon as we can find an integer $i\geqq 0$ such that if we let $n'=n-i\left(\frac{4r-\varepsilon}{3}\right)$, $g'=g-i\left(\frac{4r+9-\varepsilon}{3}\right)$, then

$$-g'+r^2+1\leqq \varrho(g',n',r)\leqq 0. \tag{3.15}$$

In fact curves with degree n', genus g', satisfying (α) and (β) (and (γ)) exist by Lemma 3.3, therefore also curves of degree n, genus g, satisfying (α) and (β) will exist by applying Lemma 3.13 i times. Since the inequalities (3.15) are equivalent to

$$\frac{3(r-1)g-3(r+1)n+6r^2+3r+3}{r-9+2\varepsilon}\leqq i\leqq\frac{3rg-3(r+1)n+3r^2+3r}{5r+\varepsilon},$$

it is clear that $i=\left[\frac{3rg-3(r+1)n+3r^2+3r}{5r+\varepsilon}\right]$ will satisfy them. □

4 Proof of Theorem 1.2

To prove the theorem, we need to recall some general facts about Brill-Noether maps. Let $C\subset\mathbb{P}^r$ be a smooth irreducible nondegenerate curve and $V\subseteqq H^0(\mathcal{O}_C(1))$ be the vector space of sections of $\mathcal{O}_C(1)$ giving the embedding. Then the cup-product map

$$\mu_0: V\otimes H^0(\omega_C(-1))\to H^0(\omega_C)$$

is called the Brill-Noether map of C.

We wish to recall here the following properties of μ_0.

(4.1) **Lemma.** *With notation as above, we have*

(i) $\ker\mu_0\cong H^1(T_{\mathbb{P}^r|C})^*$;

(ii) $h^0(T_{\mathbb{P}^r|C})\geqq(r+1)^2-1$ *and equality holds if and only if* C *is linearly normal and* μ_0 *is surjective.*

Proof. From the Euler sequence

$$0\to\mathcal{O}_C\to V^*\otimes\mathcal{O}_C(1)\to T_{\mathbb{P}^r|C}\to 0,$$

we get the exact sequence

$$0 \to H^0(\mathcal{O}_C) \to V^* \otimes H^0(\mathcal{O}_C(1)) \to H^0(T_{\mathbb{P}^r|C}) \to H^1(\mathcal{O}_C)$$
$$\xrightarrow{\varphi} V^* \otimes H^1(\mathcal{O}_C(1)) \to H^1(T_{\mathbb{P}^r|C}) \to 0.$$

Clearly φ is the dual of μ_0, hence (i). Moreover,

$$h^0(T_{\mathbb{P}^r|C}) = -1 + (r+1)h^0(\mathcal{O}_C(1)) + \dim \operatorname{coker} \mu_0 \geqq (r+1)^2 - 1$$

and equality holds if and only if $h^0(\mathcal{O}_C(1)) = r+1$ and $\operatorname{coker} \mu_0 = \{0\}$, that is, (ii). □

Proof of Theorem 1.2. Let C be a curve as in Proposition 3.9. From general deformation theory we know that (α) implies that C belongs to a unique component $W \subseteqq H_{n,g,r}$ and that if $\pi: W \to \mathcal{M}_g$ is the rational functorial map, then

$$\operatorname{codim}_{\mathcal{M}_g} \pi(W) = \dim \operatorname{coker} \{ H^0(N_C) \xrightarrow{\Phi} H^1(\omega_C^{-1}) \},$$

where the map Φ is the coboundary map associated to the exact sequence

$$0 \to \omega_C^{-1} \to T_{\mathbb{P}^r|C} \to N_C \to 0.$$

By (α) we have $\operatorname{coker} \Phi = H^1(T_{\mathbb{P}^r|C}) \cong (\ker \mu_0)^*$ (Lemma 4.1 (i)). Hence (β) and (ii) of Lemma 4.1 show that

$$\operatorname{codim}_{\mathcal{M}_g} \pi(W) = \dim \ker \mu_0 = (r+1)(g-n+r) - g = -\varrho(g,n,r). \quad \square$$

(4.2) *Remark.* The inequalities obtained in Theorem 1.2 depend on the method of proof and do not appear to have any special geometric meaning. One possible interpretation though, as can be easily seen from the proofs, is as follows. The inequality $\varrho \geqq -\frac{1}{r} g + \frac{r+1}{r}$ (respectively $\varrho \geqq -g + r^2 + 1$) expresses the numerical condition that $H_{n,g,r}$ has to satisfy in order to contain a reducible curve of type $X = X_0 \cup \ldots \cup X_i$ (respectively $Y = Y_0 \cup \ldots \cup Y_j$), where X_0 is a linearly normal nonspecial curve and $X_1, \ldots, X_i$ are disjoint rational normal curves of degree $r-1$, each meeting X_0 in $r+1$ points (respectively Y_0 is a general smoothing of X and $Y_1, \ldots, Y_j$ are disjoint rational normal curves of degree $r-1$, each meeting Y_0 in $r+2$ points). Similarly the inequality $\varrho \geqq -\frac{5r+\varepsilon}{4r+9-\varepsilon} g + f(r)$ expresses the numerical condition that $H_{n,g,r}$ has to satisfy in order to contain a reducible curve $Z = Z_0 \cup \ldots \cup Z_k$, where Z_0 is a general smoothing of Y and $Z_1, \ldots, Z_k$ are disjoint linearly normal nonspecial curves of genus $\frac{r-\varepsilon}{3}$, each meeting Z_0 in $r+4$ points. As the referee pointed out, the first inequality can also be written as $\varrho(g,n,r-1) \geqq r+1$.

(4.3) *Example.* For $b \geqq a \geqq 4$ let $W_{a,b}$ be the component of $H_{(a+b),(a-1)(b-1),3}$ whose general point represents a curve C of type (a,b) on a smooth quadric surface in $\mathbb{P}^3$. Since C is linearly normal, it follows [by S, Proposition 2.7] that μ_0 is surjective, so, from the proof of Theorem 1.2, using the fact that $W_{a,b}$ is smooth at C, we have

$$\operatorname{codim}_{\mathcal{M}_g} \pi(W_{a,b}) = \dim \operatorname{coker} \Phi = h^1(T_{\mathbb{P}^3|C}) - h^1(N_C) = \dim \ker \mu_0 - h^1(N_C)$$
$$= -\varrho - (a-3)(b-3) < -\varrho.$$

Hence $W_{a,b}$ does not have the expected number of moduli. For every n, g such that $g > \frac{1}{6}n(n-3)+1$ every component of $H_{n,g,3}$ must be a $W_{a,b}$ for some a and b, hence there is no component of $H_{n,g,3}$ with the expected number of moduli for $\varrho < -3g + 2\sqrt{24g-15} - 6$ (and $g \geqq 43$).

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References

[ACGH] Arbarello, E., Cornalba, M., Griffiths, P., Harris, J.: Geometry of algebraic curves, vol. I. Berlin Heidelberg New York: Springer 1985

[BE] Ballico, E., Ellia, P.: Bonnes petites composantes des schémas de Hilbert de courbes lisses de $\mathbb{P}^n$. C. R. Acad. Sci., Paris **306**, 187–190 (1988)

[EH] Eisenbud, D., Harris, J.: Limit linear series: basic theory. Invent. Math. **85**, 337–371 (1986)

[H] Hartshorne, R.: Algebraic geometry. Berlin Heidelberg New York: Springer 1977

[P] Pareschi, G.: Components of the Hilbert scheme of smooth space curves with the expected number of moduli. Manuscr. Math. **63**, 1–16 (1989)

[Sa] Sacchiero, G.: Fibrati normali di curve razionali dello spazio proiettivo. Ann. Univ. Ferrara, Nuova Ser., Sez. VII **27**, 33–40 (1980)

[S] Sernesi, E.: On the existence of certain families of curves. Invent. Math. **75**, 25–57 (1984)